A Problem Concerning Nonincident Points and Blocks in Steiner Triple Systems

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Abstract

In this paper, we study the problem of finding the largest possible set of s points and s blocks in a Steiner triple system of order v, such that that none of the s points lie on any of the s blocks. We prove that $s \leq (2v + 5 - \sqrt{24v + 25})/2$. We also show that equality can be attained in this bound for infinitely many values of v.

1 Introduction

This paper is a continuation of [5], where we studied the problem of finding the largest possible set of s points and s lines in a projective plane of order q, such that that none of the s points lie on any of the s lines. It was shown in [5] that $s \leq 1 + (q + 1)(\sqrt{q} - 1)$ and equality can be attained in this bound whenever q is an even power of two, by utilising certain maximal arcs in the desarguesian plane $\mathrm{PG}(2, q)$.

This problem can also be considered in other types of block designs, such as BIBDs. Suppose $(X, \mathcal{B})$ is a (v, k, λ)-BIBD. For $Y \subseteq X$ and $\mathcal{C} \subseteq \mathcal{B}$, we say that $(Y, \mathcal{C})$ is a *nonincident* set of points and blocks if $y \notin B$ for every $y \in Y$ and every $B \in \mathcal{C}$.

Maximal arcs have been studied in the setting of BIBDs (see, e.g., [4]), and it might seem plausible that maximal arcs in BIBDs might be of relevance to this problem. However, it turns out that things are a bit more complicated.

If we are going to study this problem for BIBDs, then what better place to start than with Steiner triple systems? A *Steiner triple system of order* v (or $\mathrm{STS}(v)$), is a pair $(X, \mathcal{B})$, where X is the set of v *points* and $\mathcal{B}$ is a set of $b = v(v - 1)/6$ *blocks*, such that each block contains three points and every pair of points occurs in a unique block. It is well-known that $v \equiv 1, 3 \bmod 6$ is a necessary and sufficient condition for the existence of an $\mathrm{STS}(v)$.

A *maximal arc* in an $\mathrm{STS}(v)$ consists of a subset Y of $(v + 1)/2$ points such that every block meets Y in 0 or 2 points. When $v \equiv 3, 7 \bmod 12$, $\mathrm{STS}(v)$ containing maximal arcs can easily be constructed from the standard "doubling construction" (see, for example, [2, §3.2]). The number

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of blocks disjoint from Y is

$$\frac{v(v-1)}{6} - \binom{(v+1)/2}{2} = \frac{v^2-4v+3}{24}.$$

For $v \equiv 3, 7 \bmod 12$, $v \geq 19$, it is easy to see that $(v^2-4v+3)/24 > (v+1)/2$. For these values of v, this implies that we can find $s = (v+1)/2$ points and nonincident blocks in an STS(v) that contains a maximal arc. However, it turns out that we can do better, and we will show that the optimal value of s is roughly $v - \sqrt{6v}$, for infinitely many values of v.

Define $f(v)$ to be the maximum integer s such that there exists a nonincident set of s points and s blocks in some STS(v). Equivalently, $f(v)$ is the size of the largest square submatrix of zeroes in the incidence matrix of any STS(v). We use a simple combinatorial argument to prove the upper bound $f(v) \leq (2v+5-\sqrt{24v+25})/2$. We also show that this bound is tight for infinitely many values of v. This is done by taking $\mathcal{C}$ to be the blocks in a suitably chosen subsystem of the STS(v) and letting Y be the points not in this subdesign.

2 Main Results

Theorem 1. *For any set Y of s points in an STS(v), the number of blocks disjoint from Y is at most*

$$\frac{v(v-1)+s^2-s(2v-1)}{6}.$$

Proof. Suppose that $(X, \mathcal{B})$ is an STS(v). Denote $r = (v-1)/2$; then every point occurs in r blocks in the STS(v). For a subset $Y \subseteq X$ of s points, define $\mathcal{B}_Y = \{B \in \mathcal{B} : B \cap Y \neq \emptyset\}$ and define $\mathcal{B}'_Y = \mathcal{B} \setminus \mathcal{B}_Y$. Furthermore, for every $B \in \mathcal{B}_Y$, define $B_Y = B \cap Y$, and then define $\mathcal{C} = \{B_Y : B \in \mathcal{B}_Y\}$. Observe that $\mathcal{C}$ consists of the nonempty intersections of the blocks in $\mathcal{B}$ with the set Y. Denote $c = |\mathcal{C}| = |\mathcal{B}_Y|$.

We will study the set system $(Y, \mathcal{C})$. We have the following equations:

$$\begin{aligned} \sum_{C \in \mathcal{C}} 1 &= c \\ \sum_{C \in \mathcal{C}} |C| &= rs \\ \sum_{C \in \mathcal{C}} \binom{|C|}{2} &= \binom{s}{2}. \end{aligned}$$

From the above equations, it follows that

$$\sum_{C \in \mathcal{C}} |C|^2 = s(s+r-1).$$

Now we compute as follows:

$$\begin{aligned} 0 &\leq \sum_{C \in \mathcal{C}} (|C|-2)(|C|-3) \\ &= s(s+r-1) - 5rs + 6c, \end{aligned}$$

from which it follows that

$$\begin{aligned} c &\geq \frac{s(4r+1)-s^2)}{6} \\ &= \frac{s(2v-1)-s^2)}{6}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathcal{B}'_Y| &= b-c \\ &\leq \frac{v(v-1)}{6} - \frac{s(2v-1)-s^2)}{6} \\ &= \frac{v(v-1)+s^2-s(2v-1)}{6}. \end{aligned}$$

□

Corollary 2. *If there exists a nonincident set of s points and t blocks in an STS(v), then*

$$t \leq \frac{v(v-1)+s^2-s(2v-1)}{6}.$$

Before proving our next general result, we look at a small example. In Figure 1, we graph the functions $(v(v-1)+s^2-s(2v-1))/6$ and s for $v=39$ and $s \leq v$. The point of intersection is $(26,26)$ and it is then easy to see that $f(39) \leq 26$.

In general, it is easy to compute the point of intersection of these two functions as follows:

$$\begin{aligned} \frac{v(v-1)+s^2-s(2v-1)}{6} = s \quad &\Leftrightarrow \quad s^2-(2v+5)s+v^2-v=0 \\ &\Leftrightarrow \quad s = \frac{2v+5 \pm \sqrt{24v+25}}{2}. \end{aligned}$$

Since $s < v$, the point of intersection occurs when

$$s = \frac{2v+5-\sqrt{24v+25}}{2}.$$

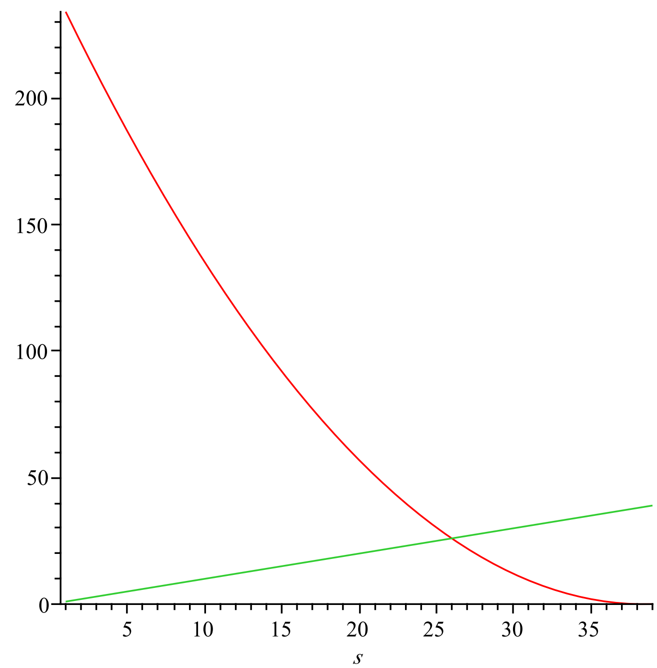
The following result is now straightforward.

Theorem 3. *For any positive integer $v \equiv 1,3 \bmod 6$, it holds that $f(v) \leq \frac{2v+5-\sqrt{24v+25}}{2}$.*

Proof. Suppose there is a nonincident set of s points and s blocks in an STS(v). Theorem 1 implies that $s \leq (v(v-1)+s^2-s(2v-1))/6$. However, for $s > (2v+5-\sqrt{24v+25})/2$, we have that $s > (v(v-1)+s^2-s(2v-1))/6$, just as in the example considered above. It follows that $s \leq (2v+5-\sqrt{24v+25})/2$. □

Next, we examine the case of equality in Theorem 1. This will involve *subsystems* of STS(v), which we now define. Suppose that $(X,\mathcal{B})$ is an STS(v). We say that $(Z,\mathcal{D})$ is a *sub-STS*(w) of $(X,\mathcal{B})$ if $Z \subset X$, $\mathcal{D} \subset \mathcal{B}$ and $(Z,\mathcal{D})$ is an STS(w). It is easy to see that an STS(v) containing a sub-STS(w) can exist only if $v \geq 2w+1$.

Figure 1: Nonincident points and lines when $v = 39$



Corollary 4. *Suppose that $(X, \mathcal{B})$ is an STS(v) and suppose we have a set $Y \subset X$ of s points such that the number of blocks in $\mathcal{B}$ disjoint from Y is equal to*

$$\frac{v(v-1) + s^2 - s(2v-1)}{6}.$$

Then $(X \backslash Y, \mathcal{B}'_Y)$ is a sub-STS$(v-s)$ of $(X, \mathcal{B})$, where $\mathcal{B}'_Y$ denotes the blocks in $\mathcal{B}$ that are disjoint from Y.

Conversely, if $(Z, \mathcal{C})$ is sub-STS(w) of $(X, \mathcal{B})$, then number of blocks in $\mathcal{B}$ disjoint from $X \backslash Z$ is equal to $(v(v-1) + s^2 - s(2v-1))/6$, where $s = |X \backslash Z| = v - w$.

Proof. From the proof of Theorem 1, it is easy to see that equality holds if and only if every block in $\mathcal{B}_Y$ meets Y in exactly 2 or 3 points. This means that no block contains two points of $X \backslash Y$, and hence $(X \backslash Y, \mathcal{B}'_Y)$ is a sub-STS$(v-s)$ of $(X, \mathcal{B})$

Conversely, suppose that $(Z, \mathcal{C})$ is sub-STS(w) of $(X, \mathcal{B})$. $\mathcal{C}$ consists of $w(w-1)/6$ blocks, and there are $v - w$ points in $X \backslash Z$. The blocks in $\mathcal{C}$ are all disjoint from $X \backslash Z$, so it suffices to verify that

$$\frac{v(v-1) + (v-w)^2 - (v-w)(2v-1)}{6} = \frac{w(w-1)}{6}.$$

This is an easy computation. $\square$

Our goal is to determine the integers v such that $f(v) = (2v + 5 - \sqrt{24v + 25})/2$, i.e., where the bound in Theorem 3 is met with equality. In view of Corollary,4, this can happen if and only

if there is an STS(v) containing a sub-STS($v-s$), where $s=(2v+5-\sqrt{24v+25})/2$. Therefore, we next determine the integers v such that the following conditions are satisfied:

1. $v \equiv 1, 3 \bmod 6$
2. $s=(2v+5-\sqrt{24v+25})/2$ is an integer
3. $v-s \equiv 1, 3 \bmod 6$, and
4. $v \geq 2(v-s)+1$.

First, condition 2. implies that $24v+25$ is a perfect square, say $24v+25=t^2$. Then we have

$$v=\frac{t^2-25}{24} \quad \text{and} \quad s=v-\frac{t-5}{2}.$$

Observe that v is an integer only when $t \equiv 1, 5 \bmod 6$.

First, suppose $t \equiv 1 \bmod 6$ and write $t=6u+1$. It is then easy to see that

$$v=\frac{3u^2+u-2}{2} \quad \text{and} \quad s=\frac{3u^2-5u+2}{2}.$$

Now, we consider requirements 1. and 3. A straightforward calculation shows that these conditions are satisfied if and only if $u \equiv 1, 5 \bmod 12$. If we let $u=12z+1$, then we get

$$v=216z^2+42z+1 \quad \text{and} \quad s=216z^2+6z, \tag{1}$$

while if $u=12z+5$, we have

$$v=216z^2+186z+39 \quad \text{and} \quad s=216z^2+150z+26. \tag{2}$$

The case $t \equiv 5 \bmod 6$ is handled in a similar way. We can write $t=6u-1$ and then we compute

$$v=\frac{3u^2-u-2}{2} \quad \text{and} \quad s=\frac{3u^2-7u+4}{2}.$$

Here it turns out that requirements 1. and 3. are satisfied if and only if $u \equiv 4, 8 \bmod 12$. If we let $u=12z+4$, then we get

$$v=216z^2+138z+21 \quad \text{and} \quad s=216z^2+102z+12, \tag{3}$$

while if $u=12z+8$, we have

$$v=216z^2+282z+91 \quad \text{and} \quad s=216z^2+246z+70. \tag{4}$$

In all four cases (1), (2), (3) and (4), it is easy to see that condition 4. is automatically satisfied. It follows that these four cases are the only situations where it is possible to have equality in Theorem 3. We now show that the desired designs exist in these cases, by making use of the following well-known result first proven in [3].

Theorem 5 (Doyen-Wilson Theorem). *There exists an STS(v) containing a sub-STS(w) if and only if $v \geq 2w+1$, $v, w \equiv 1, 3 \bmod 6$.*

The above discussion and Theorem 5 immediately imply our main existence result.

Theorem 6. *Suppose $v \equiv 1, 3 \bmod 6$ is a positive integer. Then $f(v) = \frac{2v+5-\sqrt{24v+25}}{2}$ if and only if*

$$v \in \{216z^2 + 42z + 1, 216z^2 + 186z + 39, 216z^2 + 138z + 21, 216z^2 + 282z + 91\},$$

where z is a non-negative integer.

The three smallest cases where $f(v)$ attains its optimal value are when $v = 21$, $s = 12$; $v = 39$, $s = 26$; and $v = 91$, $s = 70$.

References

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